# Mathematical Physics - B.Sc. Physics - Part-III

Dr. Sunil Kumar Yadav; email- sunil.phy30@gmail.com

In earlier notes, we have discussed —

i) General curvilinear coordinates, orthogonal curvilinear coordinates
ii) Gradient, divergence and curl in orthogonal curvilinear coordinates
iii) Laplacian in orthogonal curvilinear coordinates.
iv) Some special coordinate systems -
   (a) Plane Polar Coordinates
   (b) Right circular cylindrical coordinates
   (c) Spherical Polar Coordinates.

**Q1.** Prove that a cylindrical coordinate system is orthogonal.

The transformation relations in cylindrical coordinates are

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z. \qquad \text{——— (1)}$$

The position vector is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

or
$$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k} \qquad \text{——— (2)}$$

The tangent vectors to the $\rho$, $\phi$ and $z$ curves are given by,

$$\frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$$

$$\frac{\partial\vec{r}}{\partial z} = \hat{k}$$

For orthogonal coordinate system, the metric coefficients ~~is~~

$g_{ij} = 0$, for $i \neq j$; where $g_{ij} = \frac{\partial \vec{r}}{\partial u_i} \cdot \frac{\partial \vec{r}}{\partial u_j}$

Here $u_i$ are the coordinate surfaces.

For cylindrical coordinates these are $(u_1, u_2, u_3) \equiv (\rho, \phi, z)$

~~Now we pro~~

To show that cylindrical coordinate system is orthogonal we ~~b~~ need to show that-

$$\frac{\partial \vec{r}}{\partial \rho} \cdot \frac{\partial \vec{r}}{\partial \phi} = 0$$

$$\frac{\partial \vec{r}}{\partial \rho} \cdot \frac{\partial \vec{r}}{\partial z} = 0$$

$$\frac{\partial \vec{r}}{\partial \phi} \cdot \frac{\partial \vec{r}}{\partial z} = 0$$

Now $\frac{\partial \vec{r}}{\partial \rho} \cdot \frac{\partial \vec{r}}{\partial \phi} = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (-\rho \sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j})$

$= -\rho \cos\phi \sin\phi + \rho \cos\phi \sin\phi = 0$

Next, $\frac{\partial \vec{r}}{\partial \rho} \cdot \frac{\partial \vec{r}}{\partial z} = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot \hat{k} = 0 + 0 = 0$.

And $\frac{\partial \vec{r}}{\partial \phi} \cdot \frac{\partial \vec{r}}{\partial z} = (-\rho \sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}) \cdot \hat{k} = 0$.

Thus we see that tangent ~~thangent~~ vectors ~~$\frac{\partial}{\partial}$~~ $\frac{\partial \vec{r}}{\partial \rho}$, $\frac{\partial \vec{r}}{\partial \phi}$ and $\frac{\partial \vec{r}}{\partial z}$ to the curve $\rho$, $\phi$ and $z$ respectively are mutually perpendicular. ~~or~~ Or one can prove their unit vectors (vector divided by its magnitude) are mutually perpendicular.

H.W. Prove that the Spherical Polar Coordinate system is orthogonal.

Q.2 Determine the scale factors for cylindrical and spherical Polar Coordinates.

i) cylindrical coordinates

We have already discussed that for orthogonal coordinates square of the length of displacement $d\vec{s}$ is given by

$$ds^2 = d\vec{r}\cdot d\vec{r} = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$$

where $h_1, h_2, h_3$ are scale factors.

So first we need to obtain the $ds^2$ for spherical coordinates. There are two methods to obtain $ds^2$: First, obtain $dx$, $dy$, $dz$ and obtain $ds^2 = dx^2 + dy^2 + dz^2$.

Second – using the position vector $(\vec{r})$, obtain $d\vec{r}$ and find ~~ds~~ $ds^2 = d\vec{r}\cdot d\vec{r}$.

The transformation rule for ~~sph~~ cylindrical coordinate is given by.

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

Now
$$dx = -\rho\sin\phi\, d\phi + \cos\phi\, d\rho$$
$$dy = \rho\cos\phi\, d\phi + \sin\phi\, d\rho$$
$$dz = dz$$

$$\cancel{dx^2 + dy^2 + dz^2 = \rho^2\sin^2\phi\, d\phi}$$

Now $ds^2 = dx^2 + dy^2 + dz^2$

$$= (-\rho\sin\phi\, d\phi + \cos\phi\, d\rho)^2 + (\rho\cos\phi\, d\phi + \sin\phi\, d\rho)^2 + dz^2$$

$$ds^2 = \rho^2\sin^2\phi\,(d\phi)^2 + \cos^2\phi\,(d\rho)^2 - \cancel{2\rho\cos\phi\sin\phi\,d\rho\,d\phi}$$
$$+ \rho^2\cos^2\phi\,(d\phi)^2 + \sin^2\phi\,(d\rho)^2 + \cancel{2\rho\cos\phi\sin\phi\,d\rho\,d\phi} + dz^2$$

$$= \rho^2(d\phi)^2 + (d\rho)^2 + (dz)^2$$

or $$ds^2 = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$$

Now the scale factors are obtained as

$$\boxed{h_1 = h_\rho = 1,\ h_2 = h_\phi = \rho,\ h_3 = h_z = 1}$$

(ii) Spherical Polar Coordinates:

We use the same method for spherical polar coordinates

The transformation rules are given by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

$$dx = -r\sin\theta\sin\phi\,d\phi + r\cos\theta\cos\phi\,d\theta + \sin\theta\cos\phi\,dr$$
$$dy = r\sin\theta\cos\phi\,d\phi + r\cos\theta\sin\phi\,d\theta + \sin\theta\sin\phi\,dr$$
$$dz = -r\sin\theta\,d\theta + \cos\theta\,dr$$

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= (-r\sin\theta\sin\phi\,d\phi + r\cos\theta\cos\phi\,d\theta + \sin\theta\cos\phi\,dr)^2$$
$$+ (r\sin\theta\cos\phi\,d\phi + r\cos\theta\sin\phi\,d\theta + \sin\theta\sin\phi\,dr)^2$$
$$+ (-r\sin\theta\,d\theta + \cos\theta\,dr)^2$$

or $ds^2 = r^2\sin^2\theta\sin^2\phi\, d\phi^2 + r^2\cos^2\theta\cos^2\phi\, d\theta^2 + \sin^2\theta\cos^2\phi\, dr^2$

$- 2r^2\cos\theta\sin\theta\cos\phi\sin\phi\, d\theta d\phi - 2r\sin^2\theta\cos\phi\sin\phi\, dr d\phi$

$+ 2r\cos\theta\sin\theta\cos^2\phi\, dr d\theta$

$+ r^2\sin^2\theta\cos^2\phi\, d\phi^2 + r^2\cos^2\theta\sin^2\phi\, d\theta^2 + \sin^2\theta\sin^2\phi\, dr^2$

$+ 2r^2\cos\theta\sin\theta\cos\phi\sin\phi\, d\theta d\phi + 2r^2\sin^2\theta\cos\phi\sin\phi\, dr d\phi$

$+ 2r\cos\theta\sin\theta\sin^2\phi\, dr d\theta.$

$+ r^2\sin^2\theta\, d\theta^2 + \cos^2\theta\, dr^2 - 2r\cos\theta\sin\theta\, dr d\theta$

$= r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)\, d\phi^2 + r^2\cos^2\theta(\cos^2\phi + \sin^2\phi)\, d\theta^2$

$+ \sin^2\theta(\cos^2\phi + \sin^2\phi)\, dr^2 + 2r\cos\theta\sin\theta(\cos^2\phi + \sin^2\phi)\, dr d\theta$

$+ r^2\sin^2\theta\, d\theta^2 + \cos^2\theta\, dr^2 - 2r\cos\theta\sin\theta\, dr\, d\theta$

$= r^2\sin^2\theta\, d\phi^2 + r^2\cos^2\theta\, d\theta^2 + \sin^2\theta\, dr^2 + \cancel{2r\cos\theta\sin\theta\, dr d\theta}$

$+ r^2\sin^2\theta\, d\theta^2 + \cos^2\theta\, dr^2 - \cancel{2r\cos\theta\sin\theta\, dr\, d\theta}$

$= r^2\sin^2\theta\, d\phi^2 + r^2 d\theta^2(\cos^2\theta + \sin^2\theta) + (\sin^2\theta + \cos^2\theta)\, dr^2$

$= r^2\sin^2\theta\, d\phi^2 + r^2 d\theta^2 + dr^2$

or $ds^2 = dx^2 + dy^2 + dz^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$

Scale factors are given by

$$\boxed{\begin{aligned} h_1 &= h_r = 1 \\ h_2 &= h_\theta = r \\ h_3 &= h_\phi = r\sin\theta \end{aligned}}$$

H.W. Determine the scale factors of Elliptic cylindrical Coordinates.

Hint: The transformation rules are given by

$x = a\cosh u\cos v$, $y = a\sinh u\sin v$, $z = z$, where $u \geq 0$, $0 \leq v < 2\pi$, $-\infty < z < \infty$

[Ans $\Rightarrow h_u = h_v = a\sqrt{\sinh^2 u + \sin^2 v}$, $h_z = 1$.]